

A COUNTEREXAMPLE TO QUESTION 1 OF “A SURVEY ON THE TURAEV GENUS OF KNOTS”

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ABSTRACT. In “A survey on the Turaev genus of knots,” Champanerkar and Kofman propose several open questions. The first asks whether the polynomial whose coefficients count the number of quasi-trees of the all-A ribbon graph obtained from a diagram with minimal Turaev genus is an invariant of the knot. We answer negatively by showing a counterexample obtained from the two diagrams of 8_{21} on the KnotAtlas and KnotScape.

1. INTRODUCTION

Champanerkar and Kofman offer a very complete “survey on the Turaev genus of knots” [CK14], and we defer the reader to this short survey rather than repeat most of the background here.

In an earlier work with Stoltzfus, they define a polynomial whose coefficients count the number of quasi-trees of the all-A ribbon graph \mathbb{G} obtained from a diagram with minimal Turaev genus. This comes from the Bollobás-Riordan-Tutte polynomial $C(\mathbb{G}, X, Y, Z)$.

Proposition 1.1. [CKS07, Proposition 3.2] *Let $q(\mathbb{G}; t, Y) = C(\mathbb{G}; 1, Y, tY^{-2})$. Then $q(\mathbb{G}; t, Y)$ is a polynomial in t and Y such that*

$$(1.1) \quad q(\mathbb{G}; t) := q(\mathbb{G}; t, 0) = \sum_j a_j t^j$$

where a_j is the number of quasi-trees of genus j . Consequently, $q(\mathbb{G}; 1)$ equals the number of quasi-trees of \mathbb{G} .

Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus in [DFK⁺10, Theorem 3.2] show that the evaluation of this polynomial with $t = -1$ gives the determinant of the knot.

The recent survey paper asks whether the polynomial itself is an invariant when the Turaev genus g_T of the diagram is equal to that of the knot, that is, when it is minimal.

Question 1.2. [CK14, Question 1] Let \mathbb{G} be the all-A ribbon graph for a diagram D of a knot K . If $g_T(D) = g_T(K)$, is $q(\mathbb{G}; t)$ an invariant of K ?

We give a negative answer to this question by providing a counterexample.

Theorem 1.3. *The polynomial whose coefficients count the number of quasi-trees of the all-A ribbon graph obtained from diagram with minimal Turaev genus is not an invariant of the knot.*

We prove this Theorem 1.3 by considering the two diagrams of 8_{21} obtained from the Knot Atlas [BNMea] and KnotScape [HT99]. We address these cases in Examples 2.1 and 2.2, respectively.

We rely on an algorithm given by Armond, Druivenga, and Kindred in [ADK14] to obtain alternating diagrams on a surface with minimal Turaev genus.

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2. A COUNTEREXAMPLE: DIAGRAMS FROM THE KNOT ATLAS AND KNOTSCAPE

In Examples 2.1 and 2.2 below, we count the quasi-trees of the all-A ribbon graph obtained from diagrams coming from the Knot Atlas [BNMea] and KnotScape [HT99], respectively, as shown in Figure 1. We show that the polynomial $q(\mathbb{G}, t)$ is not invariant on the knot.



FIGURE 1. The knot 8_{21} presented in diagrams given by the Knot Atlas [BNMea] and KnotScape [HT99].

Example 2.1. Consider first the knot diagram of 8_{21} given by the Knot Atlas [BNMea], as shown in Figure 1. This diagram has Turaev genus 2. We perform a Reidemeister III move on the upper central three crossings to obtain a diagram of Turaev genus 1.

Armond, Druivenga, and Kindred [ADK14] give an algorithm to obtain an alternating diagram on a surface. We apply this to obtain a Heegaard diagram, where the dashed and dotted lines represent α and β curves, respectively, as given on the left-hand side in Figure 2.

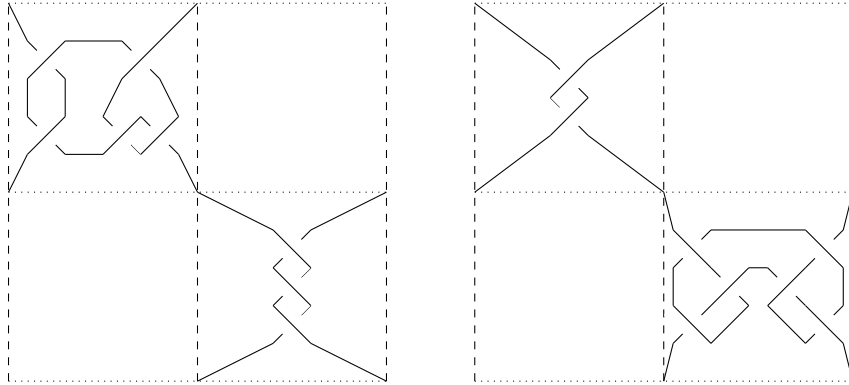


FIGURE 2. Alternating diagrams on the torus for 8_{21} coming from the KnotAtlas and KnotScape, respectively, after applying the algorithm of [ADK14].

We checkerboard color this diagram on the torus to obtain the all-A ribbon graph given on the left-hand side in Figure 3. We proceed to count the number of quasi-trees.

First of all, any spanning tree of \mathbb{G} must contain exactly two edges from the loop consisting of edges a , b , and c . From here the spanning trees fall into two classes: those with one of the two edges g and h and those with neither g nor h . Any spanning tree in the first class must contain one of the two edges d and e giving a total of $3 \times 2 \times 2 = 12$ spanning trees in the first class. Any spanning tree in the second class must contain two of the three edges d , e , and f giving a total of $3 \times 3 = 9$ spanning trees in the second class. Thus for this ribbon graph we get $a_0 = 21$.

A quasi-tree of \mathbb{G} with genus 1 must contain all of the edges a , b , and c as well as one of the two edges g and h and again these quasi-trees fall into two classes: those that contain the edges d and

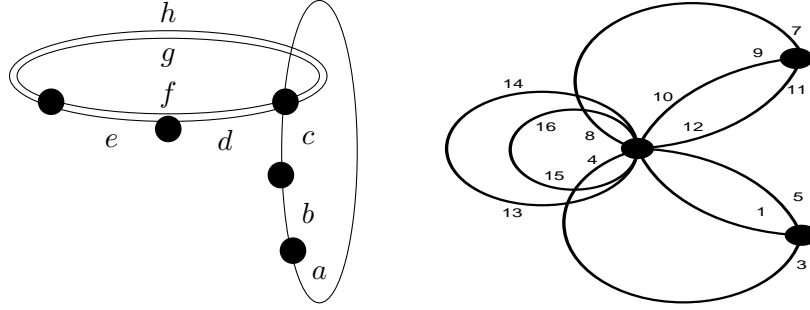


FIGURE 3. The all-A ribbon graphs for diagrams of 8_{21} coming from the KnotAtlas after a Reidemeister move III and from KnotScape (as appearing in [DFK⁺10, Figure 3]), respectively.

e but not f and those that contain the edge f and exactly one of the edges d and e . This gives us $a_1 = 4 + 2 = 6$ for this ribbon graph.

Thus, we obtain $q(\mathbb{G}, t) = 6t + 21$.

Example 2.2. Now consider the knot diagram of 8_{21} given by KnotScape [HT99] having Turaev genus 1 already and appearing on the right-hand side of Figure 1.

We apply the algorithm of [ADK14] to obtain an alternating diagram on a surface, which again is a Heegaard diagram, where the dashed and dotted lines represent α and β curves, respectively, as given on the right-hand side in Figure 2.

We checkerboard color this diagram on the torus to obtain the all-A ribbon graph, given in Figure 3 from [DFK⁺10], which we include on the right-hand side in our Figure 3.

As observed in [DFK⁺10], this ribbon graph contains 9 spanning trees and 24 genus-1 quasi-trees, yielding $q(\mathbb{G}, t) = 24t + 9$.

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